Parabolic Anderson models — Large scale asymptotics

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The Parabolic Anderson Model (PAM) is formulated in the form

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + V(t,x)u(t,x) \\ \\ u(0,x) = u_0(x) \quad x \in \mathbb{R}^d \end{cases}$$

where $\{V(t,x);\ x \in \mathbb{R}^d\}$ is a random field called potential. In this talk, $V(t,x)$ is a mean-zero stationary generalized Gaussian field with covariance function

$$\mathrm{Cov}\big(\mathrm{V(t,x), V(s,y)}\big) = \gamma_0(\mathrm{t-s})\gamma(\mathrm{x-y})$$

Throughout we assume the homogeinity

$$\gamma_0(\mathrm{cu}) = \mathrm{c}^{-\alpha_0}\gamma_0(\mathrm{u}) \ \text{ and } \ \gamma(\mathrm{cx}) = \mathrm{c}^{-\alpha}\gamma(\mathrm{x})$$

for some $\alpha_0, \alpha > 0$.

An important special case is the fractional noise (appering in most of tslk)

$$V(t,x) = \dot{W}^H(t,x) = \frac{\partial^{d+1} W^H(t, x_1, \cdots, x_d)}{\partial t \partial x_1 \cdots \partial x_d}$$

the formal derivative of the fractional Brownian sheet $W(t, x_1, \cdots, x_d)$ with the Hurst parameter $H = (H_0, H_1, \cdots, H_d)$ $(H_0, \cdots, H_d \in (0,1))$. The homogeinity holds in this case with

$$\alpha_0 = 2 - 2H_0 \;\; \text{and} \;\; \alpha = 2d - 2\sum_{j=1}^{d} H_j$$

It is known that the covarinace function in this case is

$$\mathrm{Cov}\left(\dot{W}^H(t,x), \dot{W}^H(s,y)\right) = \gamma_0(s-t)\gamma(x-y)$$

with homogenity

$$\gamma_0(cu) = c^{-\alpha_0}\gamma_0(u) \text{ and } \gamma(cx) = c^{-\alpha}\gamma(x)$$

Model on population density

In a population model.

$u(t, x)$: The particle density at the time t and the site x.

Two driving forces:

1. Birth and death: with the rate $V(t, x)$.

2. Migration: in the direction of the fastest decrease in population density: $-\nabla u(t, x)$ (Fick's law)

We show that this model is described by Anderson equation.

Given a nice bounded domain $D \subset \mathbb{R}^d$, the change rate of the population in D:

$$\frac{d}{dt}\int_D u(t,x)dx = \frac{1}{2}\int_{\partial D}(\nabla u(t,x)\cdot\vec{n})dS + \int_D V(t,x)u(t,x)dx$$

where $\vec{n} = \vec{n}(x)$ is the unit normal vector in out-bond direction.

By divergence theorem,

$$\int_{\partial D}(\nabla u(t,x)\cdot\vec{n})dS = \int_D \text{div}(\nabla u(t,x))dx = \int_D \Delta u(t,x)dx$$

So we have

$$\int_D \partial_t u(t,x)dx = \frac{1}{2}\int_D \Delta u(t,x)dx + \int_D V(t,x)u(t,x)dx$$

which leads to the Anderson equation

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + V(t,x)u(t,x)$$

Association to KPZ equation

A hot topic is the study of Kardar-Parisi-Zhang (Phy. Rev. Letters (1986)) (KPZ) equation in the case $d = 1$:

$$\partial_t h = \frac{1}{2}\Big(\frac{\partial h}{\partial x}\Big)^2 + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \dot{W}(t,x)$$

where $\dot{W}(t,x) = \frac{\partial^2}{\partial t \partial x} W(t,x)$ is an 1-dimensional time-space white noise ($H_0 = H = 1/2$).

KPZ equation describes the stochastic growth of the interface. See Martin Hairer (Ann. Math. (2013)) for mathematical treatment.

Association to KPZ equation

Under the Hopf-Cole transform

$$h(t, x) = \log u(t, x)$$

KPZ equation is formally transformed into the parabolic Anderson equation with $V(t, x) = \dot{W}(t, x)$

Large scale asymptotics

Different from the small-scale properties such as regularity of the sample path and small-time asymptotics, the large scale asymptotics concern the long term or high moment behaviors of the system, or the the pattern of the dynamics over a large space domain. There are two types of large scale asymptotics:

1. Annealed laws — the limit laws for $\mathbb{E}\, u^n(t, x)$ as $t \to \infty$ or as $m \to \infty$.

2. Quenched laws — the limit laws for $u(t, x)$ conditioning on $V(t, x)$ as what can be said as $t \to \infty$ or $|x| \to \infty$.

Mathematical definition of Pam

Take

$$V(t,x) = \dot{W}^H(t,x) = \frac{\partial^{d+1} W^H(t, x_1, \cdots, x_d)}{\partial t \partial x_1 \cdots \partial x_d}$$

The parabolic Anderson model is rigoriosly defined by the integral equation:

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) u(s,y) W^H(dyds)$$

where $p_t(x)$ is the Brownian semi-group

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{\frac{|x|^2}{2t}\right\} \quad x \in \mathbb{R}^d$$

Here we assume that the stochastic integral on the right hand side is in the Ito-Skorohkod sense.

Feymann-Kac representation

For simplicity, we take $u_0(x) = 1$ in the rest of discussion. The solvability (existance/uniqueness) of the system is largely credited to Hu and Nualart and the Kansus school and their collaborators for their work in recent years. In their work, the solution can be formally given as

$$\begin{aligned} u(t, x) =& \mathbb{E}_x \exp \left\{ \int_0^t \dot{W}^H \big(t - s, B(s)\big) ds \right. \\ & \left. - \frac{1}{2} \int_0^t \int_0^t \gamma_0(s - r) \gamma\big(B(s) - B(r)\big) ds dr \right\} \end{aligned}$$

for $m = 2, 3 \cdots$, where $B(s)$ is a d-dimensional Brownian motion independent of W^H with $B_0 = x$ and the expection $\mathbb{E}_x$ is with respect to $B(s)$.

Feymann-Kac representation

Unfortunately, the time interals in the Feymann-Kac representation do not make sense in the general case considered here. On the other hand, based on it and a formal computation one can get the rigoruous moment representation

$$\mathbb{E}\, u^m(t,x) = \mathbb{E}_0 \exp\left\{ \sum_{j<k}^{m} \int_0^t \int_0^t \gamma_0(s-r)\gamma\big(B_j(s) - B_k(r)\big) ds dr \right\}$$

where $B_1(s), \cdots, B_m(s)$ are independent d-dimensional Brownian motions with $B_j(0) = 0$. This formula becomes our starting point for the moment asymptotics.

Intermittency

The intermittency of the system can be defined by the moment asymptotic beavior

$$\lim_{t\to\infty} t^{-\beta} \log \mathbb{E}\, u^m(t,x) = \kappa(m)$$

with $\kappa(m)/m$ increases to infinity as $m \to \infty$, where $\beta > 0$ is independent of m and x. By Gärtner-Ellis's theorem, this leads to

$$\lim_{t\to\infty} t^{-\beta} \log \mathbb{P}\{\log u(t,x) \ge \lambda t^{\beta}\} = -\sup_m \{m\lambda - \kappa(m)\} \quad \lambda > 0$$

It shows that in a long run, the high peak appears at the given site x with a exponentially small but positive probability.

Intermittency

Similarly, the high moment asymptotics (if true)

$$\lim_{m\to\infty} m^{-\beta}\log \mathbb{E}\, u^m(t,x) = \kappa(t)$$

with $\beta > 2$ leads to

$$\lim_{m\to\infty} m^{-\beta}\log \mathbb{P}\Big\{ \log u(t,x) \geq m^{\beta-1}\Big\} = -\sup_{\theta>0}\Big\{\theta - \kappa(t)\theta^{\beta}\Big\}$$

or

$$\lim_{a\to\infty} a^{-\beta(\beta-1)^{-1}}\log \mathbb{P}\Big\{ \log u(t,x) \geq a\Big\}$$
$$= -(\beta-1)^{-\frac{\beta+1}{\beta-1}}(\beta-2)\kappa(t)^{-(\beta-1)^{-1}}$$

Intermittency

Theorem (C. AIHP (2017, 2019))

When $\gamma_0(\cdot) = C_0|\cdot|^{-\alpha_0}$ *with* $0 < \alpha_0 < 1$ *(i.e.,* $\dot{W}^H(t,x)$ *is colored in time) and when the* H *satisfies the condition for solvability,*

$$\lim_{t\to\infty} t^{-\frac{4-\alpha-2\alpha_0}{2-\alpha}} \log \mathbb{E}\, u^m(t,x) = \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}} m(m-1)^{\frac{\alpha}{2-\alpha}} \mathcal{E}$$

for $m = 2, 3, \cdots$*. In addition,*

$$\lim_{m\to\infty} m^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E}\, u^m(t,x) = \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}} t^{\frac{4-\alpha-2\alpha_0}{2-\alpha}} \mathcal{E}$$

Remark. Universality, super-linear growth in t.

Intermittency

In the theorem, $\mathcal{E}$ is given in terms of variation

$$\mathcal{E} = \sup_{g\in\mathcal{A}_d}\left\{\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma_0(s-r)\gamma(x-y)g^2(x)g^2(y)dxdydsdr - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d}|\nabla g(s,x)|^2dxds\right\}$$

where

$$\mathcal{A}_d = \left\{g(s,x);\ \|g(s,\cdot)\|_2 = 1 \text{ and } \|\nabla g(s,\cdot)\|_2 < \infty \text{ for } 0 \le s \le 1\right\}$$

Intermittency

When $\dot{W}^H(t,x)$ is white in time, i.e., $\gamma_0(\cdot) = \delta_0(\cdot)$ ($\alpha_0 = 1$),

$$\mathbb{E}\, u(t,x)^m = \mathbb{E}_0 \exp\left\{ \sum_{j<k}^{m} \int_0^t \gamma\big(B_j(s) - B_k(s)\big) ds \right\}$$

In this case, the variation appearing in the previous theorem becomes

$$\mathcal{E} = \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dx dy - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}$$

where

$$\mathcal{F}_d = \big\{ g(x);\ \|g\|_2 = 1 \text{ and } \|\nabla g\|_2 < \infty \big\}$$

Intermittency

Theorem (C. Ann. Probab.(2016) and AIHP(2017))

Assume $\dot{W}^H$ *is white in time. Under the solvability condition,*

$$\lim_{t\to\infty}\frac{1}{t}\log Eu^m(t,x)=\mathcal{L}(m)$$

for $m=2,3,\cdots$. *In addition,*

$$\lim_{m\to\infty} m^{-\frac{4-\alpha}{2-\alpha}}\log\mathbb{E}\,u^m(t,x)=t\Big(\frac{1}{2}\Big)^{\frac{2}{2-\alpha}}\mathcal{E}$$

where $\mathcal{L}(m)$ *is given as variation satisfyies*

$$\lim_{m\to\infty} m^{-\frac{4-\alpha}{2-\alpha}}\mathcal{L}(m)=\Big(\frac{1}{2}\Big)^{\frac{2}{2-\alpha}}\mathcal{E}$$

Remark. Linear growth in time and asymptotic universality.

Intermittency

An interesting problem is to find the exact dependence of $\mathcal{L}(m)$ on m. This problem is of significance in physics. The only case we know of is the when $\dot{W}^H$ is an one-dimensional white noise (i.e., $d = 1$ and $\gamma_0(\cdot) = \gamma(\cdot) = \delta_0(\cdot)$).

Theorem (Bertini-Cancini Statist. Phy.(1995) and C. AIHP(2015))

When $\dot{W}^H$ is an one-dimensional white noise

$$\lim_{t\to\infty} \frac{1}{t} \log E u^m(t,x) = \frac{1}{24}(m^3 - m)$$

Intermittency

Given that $\alpha = 1$ and $\mathcal{E} = 1/6$ for one-dimensional time-space white noise, the above theorem supports the following conjecture.

Conjecture. When $\dot{W}^H$ is white in time (i.e., $\gamma_0(\cdot) = \delta_0(\cdot)$),

$$\lim_{t\to\infty} \frac{1}{t} \log E u^m(t,x) = \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}} \left(m^{\frac{4-\alpha}{2-\alpha}} - m\right)\mathcal{E}$$

For comparison, recall that when $\dot{W}^H$ is colored in time ($\alpha_0 < 1$),

$$\lim_{t\to\infty} t^{-\frac{4-\alpha-2\alpha_0}{2-\alpha}} \log E u^m(t,x) = \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}} m(m-1)^{\frac{2}{2-\alpha}}\mathcal{E}$$

Intermittency

For the first time, it is shown in a recent paper (C, AIHP(2019+) that the parabolic Anderson model can be solvable when $\dot{W}^H$ is rough in time (i.e., $\alpha_0 > 1$). Another surprise finding is that the weak intermittency (C, AIHP(2019+)

$$\lim_{t\to\infty} \frac{1}{t} \log E u^2(t,x) = \kappa(2)$$

provides a linear (instead of sub-linear!) growth in time. The full intermittency and the high moment asymptotics in this case remain unknown at this time.

Quenched space asymptotics

The problem is concerned with the almost sure qrowth rate of the quantity

$$\max_{|x|\leq R} u(t,x)$$

as $R\to\infty$. The following is the first work on this regard.

Theorem (Conus, Joseph and Koshnevisan (2013), Ann. Probab.)

When $d=1$ *and* $\dot{W}^H$ *is white in time and space,*

$$0<\liminf_{R\to\infty}(\log R)^{-2/3}\log\max_{|x|\leq R} u(t,x)$$
$$\leq \limsup_{R\to\infty}(\log R)^{-2/3}\log\max_{|x|\leq R} u(t,x)<\infty \quad \text{a.s.}$$

Quenched space asymptotics

It turns out that the quenched law can be extended to a much larger class of Gaussian noise with a more precise form.

Theorem (C. Ann. Probab (2016))

When $\dot{W}^H$ *is white or colored in time and space*

$$\lim_{R\to\infty} (\log R)^{-\frac{2}{4-\alpha}} \log \max_{|x|\le R} u(t,x) = \frac{4-\alpha}{4}\left(\frac{4\mathcal{E}}{2-\alpha}\right)^{\frac{2-\alpha}{4-\alpha}} d^{\frac{2}{4-\alpha}} t^{\frac{4-\alpha-2\alpha_0}{4-\alpha}} \quad a.s.$$

When $d = 1$ *and when* $\dot{W}^H$ *is white in time and space, in particular,*

$$\lim_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x) = \frac{3}{4}\sqrt[3]{\frac{2t}{3}} \quad a.s.$$

Application to KPZ equation

Recall the one-dimensional KPZ equation

$$\partial_t h = \frac{1}{2}\Big(\frac{\partial h}{\partial x}\Big)^2 + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \dot{W}(t,x)$$

where $\dot{W}(t,x)$ is and $(1+1)$-dimensional white noise. Under the Hopf-Cole transform

$$h(t,x) = \log u(t,x)$$

the previous theorem leads to

$$\lim_{R\to\infty}(\log R)^{-2/3}\max_{|x|\le R} h(t,x) = \frac{3}{4}\sqrt[3]{\frac{2t}{3}} \quad \text{a.s.}$$

Idea of the proof

Write $\lambda_0 > 0$ for the constant appearing on the right hand side of the quenched law.

By Borel-Cantelli lemma all we need is to show

$$\sum_k \mathbb{P}\Big\{ \log \max_{|x|\le 2^k} u(t,x) \ge \lambda(\log 2^k)^{\frac{2}{4-\alpha}} \Big\} < \infty$$

for $\lambda > \lambda_0$, and

$$\sum_k \mathbb{P}\Big\{ \log \max_{|x|\le 2^k} u(t,x) \le \lambda(\log 2^k)^{\frac{2}{4-\alpha}} \Big\} < \infty$$

for $\lambda < \lambda_0$.

Idea of the proof

Given $a > 0$, set

$$Q_k = a\mathbb{Z} \cap \{|x| \le 2^k\}$$

In a suitable sense

$$\max_{|x|\le 2^k} u(t,x) \approx \max_{z\in Q_k} u(t,z)$$

when $a > 0$ is small. Under the initial condition $u_0(x) = 1$, $u(t,x)$ is stationaryin x. Hence,

$$\mathbb{P}\left\{ \log \max_{z\in Q_k} u(t,z) \ge \lambda(\log 2^k)^{\frac{2}{4-\alpha}} \right\}$$
$$\le C2^{dk}\mathbb{P}\left\{ \log u(t,0) \ge \lambda(\log 2^k)^{\frac{2}{4-\alpha}} \right\}$$

Idea of the proof

On the other hand, $u(t,z)$ $(z\in Q_k)$ are "nearly" independent when $a>0$ is large. Hence,

$$\mathbb{P}\Big\{\log\max_{z\in Q_k} u(t,z)\le\lambda(\log 2^k)^{\frac{2}{4-\alpha}}\Big\}$$

$$\approx\bigg(1-\mathbb{P}\Big\{\log u(t,0)\ge\lambda(\log 2^k)^{\frac{2}{4-\alpha}}\Big\}\bigg)^{C2^{dk}}$$

As a consequence of high moment asymptotics,

$$\mathbb{P}\Big\{\log u(t,0)\ge\big(\lambda_0\pm\epsilon\big)(\log 2^k)^{\frac{2}{4-\alpha}}\Big\}$$

$$\approx\exp\Big\{-(d\pm\delta)\log 2^k\Big\}$$

for large k.

Idea of the proof

Summarizing our steps, we have

$$\sum_k \mathbb{P}\Big\{ \log \max_{z\in Q_k} u(t,z) \geq (\lambda_0+\epsilon)(\log 2^k)^{\frac{2}{4-\alpha}} \Big\} < \infty$$

and

$$\sum_k \mathbb{P}\Big\{ \log \max_{z\in Q_k} u(t,z) \leq (\lambda_0-\epsilon)(\log 2^k)^{\frac{2}{4-\alpha}} \Big\} < \infty$$

That is what we need. □

Quenched time asymptotics

We first consider the parabolic Anderson model

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \dot{W}^H(x)u(x) \\ u(0,x) = 1 \end{cases}$$

with $\dot{W}^H(x)$ being a **time-independent** fractional Gaussian noise of the Hurst parameter $H = (H_1, \cdots, H_d)$. Recall our notation

$$\alpha = 2d - 2\sum_{j=1}^{d} H_j$$

Quenched time asymptotics

Without renormalization, we have the Feynman-Kac representation

$$u(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t \dot{W}^H(B_s)ds \right\}$$

whenever the expression on the right hand side makes reasonable sense.

Our goal is the almost sure long term asymptotics for $u(t,x)$. Carmona and Molchanov (1995, PTRF) conjectured that under the proper assumption

$$u(t,x) \sim Ct(\log t)^{\frac{4-\alpha}{2-\alpha}} \quad \text{a.s.} \qquad (t \to \infty)$$

Quenched time asymptotics

The answer is different:

Theorem (C. Ann. Probab.(2014))

Assume that $0 < \alpha < 2 \wedge d$.

$$\lim_{t\to\infty} t^{-1}(\log t)^{-\frac{2}{4-\alpha}} \log u(t,x) = \frac{4-\alpha}{4}\Big(\frac{4\mathcal{E}}{2-\alpha}\Big)^{\frac{2-\alpha}{4-\alpha}} d^{\frac{2}{4-\alpha}} t^{\frac{4-\alpha-2\alpha_0}{4-\alpha}} \quad \text{a.s.}$$

When $d = 1$ *and* $\dot{W}^H(x)$ *is white* ($\alpha = 1$)*, the above condition does not hold, but we still have*

$$\lim_{t\to\infty} t^{-1}(\log t)^{-2/3} \log u(t,x) = \frac{1}{2}\Big(\frac{3}{2}\Big)^{2/3} \quad \text{a.s.}$$

Quenched time asymptotics

Consider the case when $d = 1$ and when $\dot{W}^H(x)$ is rough, i.e., $0 < H < 1/2$ or $\alpha = 2 - 2H > 1$. In this case, the covariance

$$\gamma(x) = C_H \int_{\mathbb{R}} e^{i\xi x} |\xi|^{1-2H} d\xi$$

is interpreted as a generalized function.

Theorem (Chakrabotry, C., Gao and Tindel, Stoch. Proc. Appl. (2019+))

$$\lim_{t\to\infty} t^{-1}(\log t)^{-\frac{1}{1+H}} \log u(t,x) = \frac{1+H}{2}\left(\frac{2\mathcal{E}}{H}\right)^{\frac{1}{1+H}} \quad \text{a.s.}$$

Quenched time asymptotics

The setting of the **time-depent noise** $\dot{W}^H(t,x)$ is very different as far as the quenched long term asymptotics is concered. All evindence suggests the pattern

$$\lim_{t\to\infty}\frac{1}{t}\log u(t,x)=C_0\quad \text{a.s.}$$

In the skorohkod regime, Bertini-Giacomin (PTRF, 1999) prove that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\,\log u(t,x)=-\frac{1}{24}$$

which suggests $C_0=-1/24$. The general case still remains unknown.

Quenched time asymptotics

A much harder game is to understand the long term behavior of

$$\log u(t,x) - \mathbb{E}\log u(t,x)$$

The only case we know the answer is when $d = 1$ and $\dot{W}^H(t,x)$ is a time-space white noise,

Theorem (Amir, Corwin and Quastel Comm. Pure Appl. Math. (2011))

When $d = 1$, *and* $\dot{W}^H(t,x)$ *is a time-space white noise and when* $u_0(x) = \delta_0(x)$,

$$2^{1/3}t^{-1/3}\left(\log u(t, t^{2/3}x) + \frac{t}{24}\right) \xrightarrow{d} \mathcal{L}_{GUE}$$

where $\mathcal{L}_{GUE}$ *is the GUE Tracy-Widom distribution.*

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Thank you!